Majorants of meromorphic functions with fixed poles

A. D. Baranov, A. A. Borichev, V. P. Havin

Abstract. Let B be a meromorphic Blaschke product in the upper half-plane with zeros z_n and let $K_B = H^2 \ominus BH^2$ be the associated model subspace of the Hardy class. In other words, K_B is the space of square summable meromorphic functions with the poles at the points $\overline{z}_n$. A nonnegative function w on the real line is said to be an admissible majorant for K_B if there is a non-zero function $f \in K_B$ such that $|f| \le w$ a.e. on $\mathbb{R}$. We study the relations between the distribution of the zeros of a Blaschke product B and the class of admissible majorants for the space K_B.

Keywords. Hardy space, Blaschke product, model subspace, entire function, meromorphic function, Hilbert transform, admissible majorant.

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Introduction[1]

Let $a > 0$ and let PW_a be the Paley–Wiener space of entire functions of exponential type at most a, whose restrictions to the real axis $\mathbb{R}$ belong to $L^2(\mathbb{R})$. It is well known that the space PW_a coincides with the Fourier image of the space of square integrable functions supported in the interval $(-a, a)$.

A nonnegative function w on the real axis $\mathbb{R}$ is said to be an admissible majorant for the Paley–Wiener space PW_a if there exists a nonzero function $f \in PW_a$ such that $|f(x)| \le w(x)$ almost everywhere on $\mathbb{R}$. It is an important problem of harmonic analysis to describe the class of admissible majorants for the Paley–Wiener spaces. An obvious necessary condition is the convergence of the logarithmic integral, that is,

$$\mathcal{L}(w) = \int_{\mathbb{R}} \frac{\log^+ w^{-1}(x)}{1+x^2} dx < \infty. \tag{1}$$

A sufficient condition of the admissibility is given by the famous Beurling–Malliavin theorem [5]: *if w satisfies (1) and the function*

$$\Omega = -\log w$$

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is Lipschitz on $\mathbb{R}$, then w is an admissible majorant for any space PW_a, $a > 0$. This is a very deep result and quite a few proofs are known now (see [9, 14, 15]).

Let us also mention another, much simpler, sufficient condition: *if w is an even function decreasing on $\mathbb{R}_+ = [0, \infty)$, and $\mathcal{L}(w) < \infty$, then the majorant w is admissible for any space PW_a, $a > 0$.*

A new approach to the Beurling–Malliavin theorem was recently proposed by V.P. Havin and J. Mashreghi [10, 11]. This approach is based on the study of the Hilbert transform of the function Ω. Combined with a recent development of the same authors and F. Nazarov [12] this approach yields a new (and, probably, the shortest) proof of the Beurling–Malliavin theorem.

Another advantage of this approach is that it is applicable to a certain class of spaces of analytic functions, which generalize the Paley–Wiener spaces, namely, to the so-called model subspaces of the Hardy class.

Let Θ be an inner function in the upper half-plane $\mathbb{C}^+$, that is, a bounded analytic function such that $\lim\limits_{y\to 0+} |\Theta(x+iy)| = 1$ for almost all $x \in \mathbb{R}$ with respect to Lebesgue measure. With an inner function Θ we associate the model subspace

$$K_\Theta = H^2 \ominus \Theta H^2$$

of the Hardy class H^2 in the upper half-plane. These subspaces (and their analogs for the unit disc) play an outstanding role both in function and operator theory (see [8, 20, 21]), in particular, in the Sz.-Nagy–Foias model for contractions in a Hilbert space. It is well known that any subspace of H^2 coinvariant with respect to the semigroup of shifts $(U_t)_{t\ge 0}$, $U_t f(x) = e^{itx} f(x)$, is a K_Θ for a certain inner function Θ.

We mention two important particular cases of the model subspaces. If $\Theta(z) = \exp(iaz)$, $a > 0$, then $K_\Theta = \exp(iaz/2) PW_{a/2}$. On the other hand, if B is a Blaschke product with zeros z_n of multiplicities m_n, that is,

$$B(z) = \prod_n e^{i\alpha_n} \left(\frac{z - z_n}{z - \overline{z}_n} \right)^{m_n}$$

(here $\alpha_n \in \mathbb{R}$ and the factors $e^{i\alpha_n}$ ensure the convergence of the product), then the subspace K_B admits a simple geometrical description: it coincides with the closed linear span in L^2 of the fractions $(z - \overline{z}_n)^{-k}$, $1 \le k \le m_n$.

The approach of Havin and Mashreghi makes it possible to obtain analogs of the Beurling–Malliavin theorem for the model subspaces. We say that w is *an admissible majorant for the space* K_Θ, if there exists a nonzero function $f \in K_\Theta$ such that $|f(x)| \le w(x)$ almost everywhere on $\mathbb{R}$. The class of admissible majorants for K_Θ we denote by $\mathrm{Adm}(\Theta)$. Note that the condition $\mathcal{L}(w) < \infty$ is necessary for the inclusion $w \in \mathrm{Adm}(\Theta)$ for any inner function Θ, since

$$\int_{\mathbb{R}} \frac{\log |f(x)|}{1 + x^2} dx > -\infty$$

for any nonzero $f \in H^2$ (see, e.g., [9], p. 32–36).

In [10, 11] the authors consider mainly the case where the inner function Θ is meromorphic in the whole complex plane. Then, up to a unimodular constant, Θ is of the form

$$\Theta(z) = \exp(iaz)B(z),$$

where $a \geq 0$ and B is a Blaschke product with zeros tending to infinity. In this case there is a well-defined branch of the argument of Θ on $\mathbb{R}$, that is, there exists an increasing function φ such that $\Theta(t) = \exp(i\varphi(t))$, $t \in \mathbb{R}$. Moreover, $\Theta'(t) = i\varphi'(t)\Theta(t)$ and

$$\varphi'(t) = |\Theta'(t)| = a + 2\sum \frac{m_n \mathrm{Im}\, z_n}{|t - z_n|^2}, \quad t \in \mathbb{R}. \tag{2}$$

If B is a meromorphic Blaschke product, then the model subspace K_B may be interpreted as the space of meromorphic functions with fixed poles in the lower half-plane, which are square summable on $\mathbb{R}$.

The sufficient conditions of admissibility obtained in [10, 11] are expressed in terms of the Hilbert transform of the function Ω and the argument φ of the inner function Θ. We state some of these results in §2. The structure of the class $\mathrm{Adm}(\Theta)$ is especially well understood in two model situations. In the first case, which is considered in [11], it is assumed that the argument of the inner function grows almost linearly, that is,

$$C_1 \leq \varphi'(t) \leq C_2, \qquad t \in \mathbb{R}, \tag{3}$$

for some positive constants C_1 and C_2. In this case the class of admissible majorants essentially coincides with the class of admissible majorants for the Paley–Wiener space (we give a precise statement in §2). On the other hand, if the zeros of B are in a sense sufficiently sparse near the real axis (for example, if they are situated on the ray $\{z = iy : y > 0\}$), then there exists a "minimal" positive admissible majorant for K_B (see [10] and, also, [2]).

There is a certain gap between these two cases. The goal of this paper is to fill in this gap and to show how the class of admissible majorants $\mathrm{Adm}(B)$ depends on the distribution of zeros of the Blaschke product B. We study admissible majorants for a class of model subspaces generated by meromorphic Blaschke products with regularly distributed zeros. In particular, we consider in detail the case when the zeros lie in a strip or in a half-strip and have a power growth.

We will frequently work with interpolating Blaschke products B with zeros z_n. In this case the functions f in K_B may be characterized by the representation

$$f(z) = \sum_n \frac{c_n}{z - \overline{z}_n}, \tag{4}$$

where $\sum_n (\mathrm{Im}\, z_n)^{-1}|c_n|^2 < \infty$ (see [20]), and thus the problem reduces to the study of majorants for the series of the form (4). This representation of the elements of

K_B makes it possible to relate our problem to recent results on quasianalyticity [16] and weighted polynomial approximation [6].

In what follows we make use of the following notations: given nonnegative functions g and h we write $g \asymp h$ if $C_1 h \le g \le C_2 h$ for some positive constants C_1 and C_2 and for all admissible values of the variables. Letters C, C_1, etc. will denote various constants which may change their values in different occurrences.

§1. Main results

As in [10, 11], in the present paper we restrict ourselves to the case of meromorphic inner functions. Let us consider the following example. Let B be the Blaschke product with simple zeros $z_n = n + i$, $n \in \mathbb{Z}$. Then, clearly, the argument of B satisfies (3). Moreover, it is easy to show (see §2) that $\mathrm{Adm}(B) = \mathrm{Adm}(e^{2\pi i z})$. Now let us take only a half of the zeros: consider the Blaschke product B_1 with zeros at the points $z_n = n + i$, $n \in \mathbb{N}$. It is a natural question whether there is a qualitative difference between the classes $\mathrm{Adm}(B_1)$ and $\mathrm{Adm}(B)$. Clearly, $\mathrm{Adm}(B_1) \subset \mathrm{Adm}(B)$, since $K_{B_1} \subset K_B$.

The following theorem answers this question. To state it let us introduce the notion of a one-sided majorant, which is natural, since the zeros are asymmetric. We say that $w \in \mathrm{Adm}_+(\Theta)$ ($w \in \mathrm{Adm}_-(\Theta)$) if there exists a nonzero function $f \in K_\Theta$ such that $|f(x)| \le w(x)$ for a.e. $x > 0$ (respectively for a.e. $x < 0$).

Theorem 1.1. *Let w be a nonnegative function on $[0,\infty)$ and let $\Omega = -\log w$.*

1. If $w \in \mathrm{Adm}_+(B_1)$, then

$$\int_1^\infty t^{-3/2}\Omega(t)dt < \infty. \tag{5}$$

2. If w is positive, nonincreasing and the integral (5) converges, then $w \in \mathrm{Adm}_+(B_1)$ and $w(|x|) \in \mathrm{Adm}(B_1)$.

It turns out that there is more freedom in the behavior of the elements of K_{B_1} along the negative semiaxis (that is, when we are far from the poles). In particular, the majorants $w(t) = \exp(-A|t|^{1/2})$ are in $\mathrm{Adm}_-(B_1)$ (whereas, by Theorem 1.1, the majorant of the form $\exp(-|t|^\alpha)$ belongs to $\mathrm{Adm}_+(B_1)$ if and only if $\alpha < 1/2$). Moreover, this result is sharp.

Theorem 1.2. *The majorant $w(t) = \exp(-A|t|^{1/2})$ belongs to the class $\mathrm{Adm}_-(B_1)$ for any $A > 0$. At the same time, if $|t|^{1/2} = o(\Omega(t))$ when $t \to -\infty$, then $w \notin \mathrm{Adm}_-(B_1)$.*

These results remain true for a wider class of Blaschke products with zeros in a half-strip having certain density properties. Let all z_n lie in a half-strip $[0,\infty)\times[\delta, M]$, where $M > \delta > 0$. Assume that

$$0 < c \le \frac{\mathrm{card}\{n : \mathrm{Re}\, z_n \in [x, x+r]\}}{r} \le C < \infty, \qquad r > r_0, \quad x > 0. \tag{6}$$

Theorem 1.3. *Let B be a Blaschke product with zeros z_n satisfying (6). Then all statements of Theorems 1.1 and 1.2 are true for B instead of B_1.*

Now we consider a class of Blaschke products with power growth of zeros. Let $\beta > 1/2$ and let B_β be the Blaschke product with simple zeros at the points

$$z_n = n^\beta + i, \quad n \in \mathbb{N}$$

(if $\beta \le 1/2$, then the Blaschke condition does not hold). Thus, the notation B_1 for the product in Theorems 1.1 and 1.2 agrees with this new notation. We will study the asymptotic decay of admissible majorants for K_{B_β}. Namely, let

$$w_\alpha(x) = \exp(-|x|^\alpha), \qquad \alpha \in (0,1),$$

and put

$$\alpha(\beta) = \sup\{\alpha : w_\alpha \in \mathrm{Adm}(B_\beta)\}.$$

Analogously, one can define the numbers $\alpha_+(\beta)$ and $\alpha_-(\beta)$. It follows from Theorems 1.1 and 1.2 that $\alpha(1) = \alpha_+(1) = \alpha_-(1) = 1/2$. The following theorem shows that even such a rough characteristic as $\alpha(\beta)$ has a rather complicated behavior.

Theorem 1.4.

$$\alpha(\beta) = \alpha_+(\beta) = \begin{cases} 1/\beta, & \beta > 2, \\ 1/2, & 2/3 \le \beta \le 2, \\ -1 + 1/\beta, & 1/2 < \beta < 2/3; \end{cases} \tag{7}$$

$$\alpha_-(\beta) = \begin{cases} 1/\beta, & \beta > 2, \\ 1/2, & 1 \le \beta \le 2, \\ 1, & 1/2 < \beta < 1. \end{cases} \tag{8}$$

Remarks. 1. It is natural to ask whether $w_{\alpha(\beta)}$ is admissible for K_{B_β}. Our argument in Section 5 shows that for $\beta > 2$ and some $\kappa > 0$ the majorant $w^\kappa_{\alpha(\beta)}$ is admissible for K_{B_β}. If $2/3 < \beta \le 2$, then, by Theorem 3.1, $w^\kappa_{\alpha(\beta)} \notin \mathrm{Adm}(B_\beta)$ for any $\kappa > 0$. Our question remains open for $\beta \in (1/2, 2/3)$.

2. An interesting feature of the limit exponent $\alpha(\beta)$ is that it is constant for $\beta \in [2/3, 2]$, though for $\beta > 1$ the zeros are sparse, whereas in the case $\beta < 1$ the zeros are much more dense (in particular, the sequence is not an interpolating one). An analogous phenomenon may be observed in the problems of weighted polynomial approximation on discrete subsets of the line (see §5). These problems turn out to be closely related to admissibility conditions.

3. It should be noted that in the case $\beta < 1$ the admissible majorants on the negative semiaxis may decrease much faster than on the positive one. However, any

nonzero function $f \in K_{B_\beta}$, $2/3 < \beta < 1$, which is majorized on $(-\infty, 0]$ by w_α with $\alpha \in (1/2, 1)$, is unbounded on $[0, \infty)$ (see Remark 5.5).

4. Analogous results are obtained for the Blaschke products with two-sided zeros having different power growth in positive and negative directions (see Theorem 5.6). We mention also that all these results may be easily generalized to the case of perturbed zeros with certain density properties.

Finally we consider the case of zeros which approach the real axis tangentially. Let B be the Blaschke product with the zeros $z_n = n + iy_n$, where $0 < y_n \le 1$, $n \in \mathbb{Z}$ (or $n \in \mathbb{N}$). The most interesting situation is when $y_n \to 0$, $|n| \to \infty$. In this case another phenomenon occurs. If y_n tend to zero not too rapidly, then there is no qualitative difference between the classes $\mathrm{Adm}(B)$ and $\mathrm{Adm}(e^{iaz})$, $a > 0$. Otherwise, there are no admissible majorants with more than power decay.

Theorem 1.5. *Let the sequence $\{y_n\}_{n\in\mathbb{Z}}$ be even and nonincreasing for $n \ge 0$.*

1. If

$$\sum_{n\in\mathbb{N}} n^{-2} \log \frac{1}{y_n} < \infty, \tag{9}$$

then any even majorant w, nonincreasing on $[0, \infty)$ and such that $\mathcal{L}(w) < \infty$, is admissible for K_B.

2. Let $y : \mathbb{R} \to (0, \infty)$ be an even function nonincreasing on $[0, \infty)$ and such that $y(n) = y_n$, $n \in \mathbb{Z}$, and let $Y = -\log y$. If the function $Y(e^x)$ is convex on $\mathbb{R}$ and the series (9) diverges, then any majorant w such that, for any $N > 0$, $w(t) = o(|t|^{-N})$, $t \to \infty$, is not admissible for K_B.

To obtain these results we combine the admissibility conditions and techniques of [10, 11] with some results on quasianalyticity.

The paper is organized as follows. In §2 we present a few results of [10, 11] and some corollaries which we use later on. In §3 we prove Theorem 1.1 whereas §4 is devoted to the proof of Theorems 1.2 and 1.3. Limit exponents for the Blaschke products with power growth of the zeros will be considered in §5. Finally, in §6 we discuss the case of tangential zeros.

§2. General sufficient conditions

Here we present some of the results of the papers [10, 11, 3]. We start with the following general criterion of admissibility. Here Θ is an arbitrary, not necessarily meromorphic, inner function. We denote by Π the Poisson measure, that is, $d\Pi(t) = \frac{dt}{t^2+1}$. If $\Omega = -\log w \in L^1(\Pi)$, then there exists the Hilbert transform of the function Ω defined as follows:

$$\widetilde{\Omega}(x) = v.p.\, \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) \Omega(t)\, dt.$$

Theorem 2.1. *A majorant w with $\Omega \in L^1(\Pi)$ is admissible for K_Θ if and only if there exists a function $m \in L^\infty(\mathbb{R})$ with $m \geq 0$, $mw \in L^2(\mathbb{R})$, and $\log m \in L^1(\Pi)$, and an inner function I such that*

$$\arg\Theta + 2\widetilde{\Omega} = 2\widetilde{\log m} + \arg I + 2\pi k \qquad a.e.\ on\ \mathbb{R}, \tag{10}$$

where k is a measurable function with integer values. Here $\arg\Theta$ is an arbitrary measurable branch of the argument.

A certain refinement of this parametrization formula for admissible majorants is obtained in [3]. Namely, it is shown that the theorem remains true if we replace $\arg I$ in (10) by a constant $\gamma \in \mathbb{R}$.

The following theorem provides a condition sufficient for the representation of the form (10). We denote by $\mathrm{Osc}(f, I)$ the oscillation of a function f on the set I, that is,

$$\mathrm{Osc}(f, I) = \sup_{s,t\in I}(f(s) - f(t)).$$

Theorem 2.2. *Let f be a C^1-function on $\mathbb{R}$ and let $\{d_n\}$ (where $n \in \mathbb{Z}$ or $n \in \mathbb{N}$; in the latter case we assume that $d_1 = -\infty$, and we do not require f to have a limit at infinity) be an increasing sequence of real numbers such that $\lim_{|n|\to\infty} |d_n| = \infty$ and*

$$f(d_{n+1}) - f(d_n) \asymp 1, \qquad n \in \mathbb{Z}\ \ (n \geq 2).$$

Assume also that there is a constant $C > 0$ such that

$$\mathrm{Osc}(f, (d_n, d_{n+1})) \leq C \qquad and \qquad \mathrm{Osc}(f', (d_n, d_{n+1})) \leq C$$

for all $n \in \mathbb{Z}$ ($n \in \mathbb{N}$). Then f admits the representation

$$f = 2\widetilde{\log m} + 2\pi k + \gamma,$$

where $m \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, $m \geq 0$, $\log m \in L^1(\Pi)$, $\gamma \in \mathbb{R}$, and k is a measurable integer-valued function.

This theorem is proved in [11] under a small additional restriction on the distances $d_{n+1} - d_n$ and in [3] in the general case. Such functions f will be referred to as *mainly increasing functions*. It should be mentioned that the condition $\mathrm{Osc}(f', (d_n, d_{n+1})) \leq C$ may be replaced by a weaker integral estimate.

Combining Theorems 2.1 and 2.2 we arrive at the following sufficient condition.

Corollary 2.3. *Let Θ be a meromorphic inner function and let φ be an increasing branch of the argument of Θ. If $\varphi + 2\widetilde{\Omega}$ is a mainly increasing function, then $w \in \mathrm{Adm}(\Theta)$.*

We have the following useful corollary.

Corollary 2.4. *Let Θ_1 and Θ_2 be meromorphic inner functions with arguments φ_1 and φ_2 respectively. If the function $\varphi_1 - \varphi_2$ is mainly increasing, then* $\mathrm{Adm}(\Theta_2) \subset \mathrm{Adm}(\Theta_1)$.

Remark. Clearly, under conditions of Corollary 2.4 analogous inclusions take place for the classes $\mathrm{Adm}_+(\Theta_j)$ and $\mathrm{Adm}_-(\Theta_j)$, $j = 1, 2$; namely, $\mathrm{Adm}_+(\Theta_2) \subset \mathrm{Adm}_+(\Theta_1)$ and $\mathrm{Adm}_-(\Theta_2) \subset \mathrm{Adm}_-(\Theta_1)$. Indeed, if $w \in \mathrm{Adm}_+(\Theta_2)$, then there exists a nonzero function $f \in K_{\Theta_2}$ such that $|f(t)| \le w(t)$, $t < 0$. Hence, $|f| \in \mathrm{Adm}(\Theta_2)$ and, by Corollary 2.4, $|f| \in \mathrm{Adm}(\Theta_1)$. Thus, $|f| \in \mathrm{Adm}_+(\Theta_1)$ and, consequently, $w \in \mathrm{Adm}_+(\Theta_1)$.

We get immediately the following corollary concerning the inner functions with "almost linear" growth of the argument.

Corollary 2.5. *Let Θ be a meromorphic inner function such that $\varphi' \asymp 1$. Then there exist positive numbers a, b such that* $\mathrm{Adm}(e^{iaz}) \subset \mathrm{Adm}(\Theta) \subset \mathrm{Adm}(e^{ibz})$.

Proof. Note that $\psi(t) = bt$ is the continuous argument of the inner function e^{ibz}. Let $c \le \varphi'(t) \le C$, $t \in \mathbb{R}$, for some positive constants c, C. Take $b > C$. Then $bt - \varphi(t)$ is an increasing Lipschitz function and, consequently, is mainly increasing. Hence, by Corollary 2.4, $\mathrm{Adm}(\Theta) \subset \mathrm{Adm}(e^{ibz})$. The proof of the second inclusion is analogous. ◯

Now we discuss another approach to admissible majorants applicable to the case of Blaschke products with sparse zeros. Let Θ be a meromorphic inner function with zeros z_n repeated according to their multiplicities. Then there exists an entire function E in the Hermite–Biehler class (that is, $|E(z)| > |E(\overline{z})|$, $z \in \mathbb{C}^+$) with zeros at the points $\overline{z}_n$ such that $\Theta = E^*/E$ (see, e.g., [10, Lemma 2.1]). Here $E^*(z) = \overline{E(\overline{z})}$.

With the function E we associate the de Branges space $\mathcal{H}(E)$ which consists of all entire functions F such that F/E and F^*/E belong to the Hardy class H^2. The book of L. de Branges [7] is devoted to the theory of spaces $\mathcal{H}(E)$; this theory has important applications in mathematical physics.

It is easy to see that the mapping $F \mapsto F/E$ is a unitary operator from $\mathcal{H}(E)$ onto K_{Θ_E}, where $\Theta_E = E^*/E$, that is, $K_{\Theta_E} = \mathcal{H}(E)/E$ (see, for example, [1] or [10, Theorem 2.10]).

An admissible majorant w is said to be minimal if for any other admissible majorant $\tilde{w}$ such that $\tilde{w} \le Cw$ we have $\tilde{w} \asymp w$, that is, $cw \le \tilde{w} \le Cw$ for some positive constant c. It turns out that the inclusion $1 \in \mathcal{H}(E)$ is crucial for the existence of a positive and continuous minimal majorant. Namely, the following dichotomy is true (see [2, 3, 10]).

Theorem 2.6. *Let E be an entire function of zero exponential type such that $|E(z)| > |E(\overline{z})|$, $z \in \mathbb{C}^+$. Then, either*

a) $1/E \in L^2(\mathbb{R})$ and $1/|E|$ is (the unique up to equivalence) positive and continuous minimal majorant for K_{Θ_E};

b) $1/E \notin L^2(\mathbb{R})$ *and there is no positive and continuous minimal majorant for* K_{Θ_E}.

To conclude this section we show that $\mathrm{Adm}(B) = \mathrm{Adm}(e^{2\pi i z})$ if B is the Blaschke product with the zeros $z_n = n + i$, $n \in \mathbb{Z}$. Put $E_1(z) = e^{i\pi z}$ and $E_2(z) = \sin \pi(z+i)$. Then $B = E_2^*/E_2$. Clearly, $|E_1(z)| \asymp |E_2(z)|$ for $\mathrm{Im}\, z \ge 0$ and so the spaces $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ coincide as sets with equivalence of norms. Since $K_{E_1^*/E_1} = \mathcal{H}(E_1)/E_1$ and $K_{E_2^*/E_2} = \mathcal{H}(E_2)/E_2$ it follows that $\mathrm{Adm}(E_1^*/E_1) = \mathrm{Adm}(E_2^*/E_2)$.

§3. Proof of Theorem 1.1

We start with the proof of Statement 1 of Theorem 1.1. Moreover, we prove a somewhat stronger result. Recall that B_β is the Blaschke product with the zeros $z_n = n^\beta + i$, $n \in \mathbb{N}$.

Theorem 3.1. *Let* $\beta > 2/3$. *If* $f \in K_{B_\beta}$, *then*

$$\int_0^\infty \frac{\log |f(t)|}{t^{3/2} + 1} dt > -\infty.$$

To prove Theorem 3.1 we make use of certain properties of model subspaces. Recall that the function

$$k_z(\zeta) = \frac{i}{2\pi} \cdot \frac{1 - \overline{\Theta(z)}\Theta(\zeta)}{\zeta - \overline{z}}$$

is the reproducing kernel of the space K_Θ corresponding to a point $z \in \mathbb{C}^+$, that is, $f(z) = (f, k_z)$, $f \in K_\Theta$, where $(\cdot, \cdot)$ stands for the usual inner product in $L^2(\mathbb{R})$. In the case of meromorphic inner functions the same formula gives the reproducing kernel at the point $z = x \in \mathbb{R}$.

Clearly,

$$|f(z)| \le \|f\|_2 \|k_z\|_2, \qquad z \in \mathbb{C}^+ \cup \mathbb{R}$$

(here we denote by $\mathbb{C}^+ \cup \mathbb{R}$ the closed upper half-plane). Note also that

$$\|k_z\|_2^2 = \frac{1 - |\Theta(z)|^2}{4\pi \mathrm{Im}\, z}, \quad z \in \mathbb{C}^+, \quad \text{and} \quad \|k_x\|_2^2 = \frac{|\Theta'(x)|}{2\pi}, \quad x \in \mathbb{R}.$$

Now let $B = \prod_n b_n$ be a Blaschke product, where $b_n(z) = e^{i\alpha_n}(z - z_n)/(z - \overline{z}_n)$. Clearly, $1 - |B(z)|^2 \le \sum_n (1 - |b_n(z)|^2)$, $z \in \mathbb{C}^+$, and hence

$$\frac{1 - |B(z)|^2}{2\mathrm{Im}\, z} \le \sum_n \frac{1 - |b_n(z)|^2}{2\mathrm{Im}\, z} = \sum_n \frac{2\mathrm{Im}\, z_n}{|z - \overline{z}_n|^2} \le |B'(\mathrm{Re}\, z)|, \quad z \in \mathbb{C}^+$$

(see (2)). Thus, we get the following estimate: if $f \in K_B$, then

$$|f(z)| \le C(f)|B'(x)|^{1/2}, \qquad z = x + iy \in \mathbb{C}^+ \cup \mathbb{R}. \tag{11}$$

We will also need to estimate the function $f \in K_B$ in the lower half-plane. It follows from the definition of K_B that the inclusion $f \in K_B$ implies that $f/B \in H^2(\mathbb{C}^-)$. Moreover, the function $g(z) = \overline{f(\overline{z})}B(z)$, $z \in \mathbb{C}^+$, is in K_B. Hence, applying the estimate (11) we have

$$|f(z)| \le C(f)|B(z)| \cdot |B'(x)|^{1/2}, \qquad z = x - iy \in \mathbb{C}^- \tag{12}$$

(here the right-hand side may be infinite).

Consider the domain

$$\Delta = \mathbb{C} \setminus \{z : \operatorname{Re} z \ge 0, -2 \le \operatorname{Im} z \le 0\}. \tag{13}$$

Then each function $f \in K_{B_\beta}$ is analytic in Δ. Let η be a conformal mapping of the upper half-plane $\mathbb{C}^+$ onto the domain Δ such that $\eta(0) = 0$, $\eta(\infty) = \infty$. By the Christoffel–Schwarz formula, η is of the form

$$\eta(z) = a_1 + a_2 \int_{z_0}^{z} \zeta^{1/2}(\zeta - a)^{1/2} d\zeta \tag{14}$$

where $a_1 \in \mathbb{C}$, $a_2 > 0$, $z_0 \in \mathbb{C}^+$ and $a > 0$. Clearly,

$$\eta(z) \sim a_2 z^2/2, \qquad \text{and} \qquad \eta'(z) \sim a_2 z$$

when $|z| \to \infty$, $z \in \mathbb{C}^+ \cup \mathbb{R}$ (we write $f(z) \sim g(z)$, $z \to z_0$, if $\lim_{z \to z_0} f(z)/g(z) = 1$).

In the proof of Theorem 3.1 we will use the following lemma.

Lemma 3.2. *Let $f \in K_{B_\beta}$. If $\beta \ge 1$, then f is bounded in Δ. If $1/2 < \beta < 1$, then*

$$|f(z)| \le C_1 \exp(C_2|z|^{-1+1/\beta}), \qquad z \in \Delta. \tag{15}$$

Proof. Since

$$|B'_\beta(x)| = \sum_{n=1}^{\infty} \frac{2}{(x - n^\beta)^2 + 1}, \tag{16}$$

it is easy to see that $B'_\beta \in L^\infty(\mathbb{R})$ for $\beta \ge 1$ and $B'_\beta \in L^\infty((-\infty, 0])$ for $1/2 < \beta < 1$. Let us show that for $1/2 < \beta < 1$

$$|B'_\beta(x)| \asymp x^{-1+1/\beta}, \qquad x > 1.$$

Let $x = t^\beta$. First we note that

$$\sum_{n=1}^{[t/2]} \frac{1}{(t^\beta - n^\beta)^2 + 1} + \sum_{n=[3t/2]}^{\infty} \frac{1}{(t^\beta - n^\beta)^2 + 1} \le Ct^{1-2\beta} = Cx^{-2+1/\beta}.$$

We denote by [s] the entire part of s. If $[t/2] \le n \le [3t/2]$, then $|t^\beta - n^\beta| \asymp |t-n| t^{\beta-1}$. Then

$$\sum_{[t/2]}^{[3t/2]} \frac{1}{(t^\beta - n^\beta)^2 + 1} \asymp \sum_{[t/2]}^{[3t/2]} \frac{1}{(t-n)^2 t^{2\beta-2} + 1}.$$

Clearly,

$$\sum_{[t]-[t^{1-\beta}]}^{[t]} \frac{1}{(t-n)^2 t^{2\beta-2} + 1} \le C t^{1-\beta}.$$

On the other hand,

$$\sum_{[t/2]}^{[t]-[t^{1-\beta}]} \frac{1}{(t-n)^2 t^{2\beta-2} + 1} \asymp t^{2-2\beta} \sum_{[t/2]}^{[t]-[t^{1-\beta}]} \frac{1}{(t-n)^2} \asymp t^{1-\beta}.$$

The estimate of the sum over $[t] < n \le [3t/2]$ is analogous.

Now it follows from (11) that $|f(z)| \le C$, $z \in \mathbb{C}^+ \cup \mathbb{R}$ in the case $\beta \ge 1$ whereas for $1/2 < \beta < 1$ we have $|f(z)| \le C_1 + C_2 |z|^{-1/2 + 1/(2\beta)}$. To estimate $|f(z)|$ for $z \in \Delta \cap \mathbb{C}^-$, it suffices to estimate $|B_\beta(z)|$ from above.

Let $z = x - iy \in \Delta \cap \mathbb{C}^-$. Then

$$2 \log |B_\beta(z)| = \sum_{n=1}^{\infty} \log \left(1 + \frac{4y}{|z - \overline{z}_n|^2} \right) \le 4y \sum_{n=1}^{\infty} \frac{1}{(x - n^\beta)^2 + (y-1)^2}.$$

By the estimates analogous to those above it is easily shown that

$$\log |B_\beta(z)| \le \begin{cases} C, & \beta \ge 1, \\ C_1 + C_2 |z|^{-1+1/\beta}, & 1/2 < \beta < 1. \end{cases} \tag{17}$$

To complete the proof of the lemma we apply estimate (12). ○

We will also make use of the following simple lemma (see [13, IIIG2]).

Lemma 3.3. *Let g be a nonzero function analytic in $\mathbb{C}^+$ and continuous in $\mathbb{C}^+ \cup \mathbb{R}$. If $\log |g(z)| \le C|z|^\gamma$, $z \in \mathbb{C}^+ \cup \mathbb{R}$, where $0 < \gamma < 1$, then*

$$\int_{\mathbb{R}} \frac{\log |g(t)|}{t^2 + 1} dt > -\infty.$$

Proof of Theorem 3.1. Let $\beta > 2/3$ and $f \in K_{B_\beta}$. By Lemma 3.2, f is bounded in Δ if $\beta \ge 1$ and $|f(z)| \le C_1 \exp(C_2 |z|^{-1+1/\beta})$, $z \in \Delta$, if $2/3 < \beta < 1$.

Put $g(z) = f(\eta(z))$, $z \in \mathbb{C}^+$, where the conformal mapping η of $\mathbb{C}^+$ onto Δ is defined by (14). Then $g \in H^\infty$ if $\beta \ge 1$ and

$$|g(z)| \le C_3 \exp(C_4 |z|^{-2+2/\beta}), \qquad z \in \mathbb{C}^+,$$

if $2/3 < \beta < 1$. Since $\beta > 2/3$ we have $-2 + 2/\beta < 1$. Hence, by Lemma 3.3,

$$\int_{\mathbb{R}} \frac{\log|g(t)|}{t^2+1} dt = \int_{\mathbb{R}} \frac{\log|f(\eta(t))|}{t^2+1} dt > -\infty.$$

Taking into account that $\eta(t) \asymp t^2$ and $\eta'(t) \asymp t$, $t \in \mathbb{R}$, $t \to +\infty$, we obtain

$$\int_0^\infty \frac{\log|f(t)|}{t^{3/2}+1} dt > -\infty. \qquad \bigcirc$$

Now we turn to the proof of Statement 2 (sufficiency) of Theorem 1.1. In what follows we will make use of the following lemma.

Lemma 3.4. *Let $\Omega = \log \frac{1}{w}$ be a nondecreasing function on $[0,\infty)$. If $f \in H^2$, $|f| \le 1$ a.e. on $\mathbb{R}$ and $|f| \le w$ a.e. on $(0,\infty)$, then*

$$\log|f(z)| \le -\frac{1}{4}\Omega(|z|), \qquad z \in \mathbb{C}^+, \quad \operatorname{Re} z > 0.$$

Proof. Let $z = x + iy \in \mathbb{C}^+$ and $x > 0$. By the Jensen inequality

$$\log|f(z)| \le \frac{y}{\pi} \int_{\mathbb{R}} \frac{\log|f(t)|}{|t-z|^2} dt.$$

Since $|f(t)| \le 1$ we have

$$\log|f(z)| \le -\frac{y}{\pi} \int_{x+y}^\infty \frac{\Omega(t)}{|t-z|^2} dt \le -\frac{\Omega(|z|)}{\pi} \int_{x+y}^\infty \frac{y dt}{(t-x)^2+y^2} dt = -\frac{1}{4}\Omega(|z|). \qquad \bigcirc$$

Recall that a sequence $\lambda_n \subset \mathbb{C}^+$ is said to be an interpolating sequence if it satisfies the Carleson condition

$$\inf_n \prod_{k \ne n} \left| \frac{\lambda_n - \lambda_k}{\lambda_n - \overline{\lambda}_k} \right| = \inf_n 2\operatorname{Im}\lambda_n |B'(\lambda_n)| > 0,$$

where B is the Blaschke product with the zeros λ_n. By the Shapiro–Shields theorem (see [20, 21]), in this case the rational fractions $k_n(z) = \frac{\sqrt{\operatorname{Im}\lambda_n}}{z - \overline{\lambda}_n}$ form a Riesz basis in K_B and, thus, the inclusion $f \in K_B$ is equivalent to the representation

$$f(z) = \sum_n \frac{c_n \sqrt{\operatorname{Im}\lambda_n}}{z - \overline{\lambda}_n}, \qquad \{c_n\} \in \ell^2, \tag{18}$$

with $\|f\|_2 \asymp \|\{c_n\}\|_{\ell^2}$.

Clearly, the Blaschke product B_1 is interpolating. Hence, each function f in K_{B_1} admits the representation

$$f(z) = \sum_{n\in\mathbb{N}} \frac{c_n}{z-n+i}. \tag{19}$$

The following two lemmas relate the rate of decay of a function $f \in K_{B_1}$ with the properties of the coefficients c_n in (19). Lemma 3.6 plays the key role in the proof of Statement 2 of Theorem 1.1. However, for the sake of completeness we start with the converse result.

Lemma 3.5. *Let w be a positive nonincreasing function on $[0,\infty)$, which tends to zero faster than any power, that is,*

$$\lim_{t\to\infty} t^N w(t) = 0 \tag{20}$$

for any $N > 0$. If $w \in \mathrm{Adm}_+(B_1)$, then there exists a nonzero sequence $\{c_n\}_{n\in\mathbb{N}}$ such that

$$\log|c_n| \le -C\Omega(n) \tag{21}$$

and all the moments of the sequence $\{c_n\}$ are equal to zero, that is,

$$\sum_{n=1}^{\infty} c_n n^k = 0, \qquad k \in \mathbb{Z}_+. \tag{22}$$

In the converse statement we impose certain regularity conditions on the majorant w. We say that a majorant w is *regular* if w is even, $0 < w \le 1$, w is nonincreasing on $[0,\infty)$ and the function $t\Omega'(t)$ is nondecreasing on $[0,\infty)$ (recall that $\Omega = \log\frac{1}{w}$). The last property is equivalent to the following: the function $G(s) = \Omega(e^s)$ is convex.

Lemma 3.6. *Let w be an even regular majorant on $\mathbb{R}$, which tends to zero faster than any power. If there exists a nonzero sequence $\{c_n\}_{n\in\mathbb{N}}$ such that*

$$|c_n| \le n^{-3} w(n), \qquad n \in \mathbb{N},$$

and equalities (22) hold, then $w \in \mathrm{Adm}(B_1)$.

Proof of Lemma 3.5. Let f be a function of the form (19) such that $|f(t)| \le w(t)$, $t > 0$. By Lemma 3.2, $K_{B_1} \subset L^\infty(\mathbb{R})$. Thus, without loss of generality we may assume that $|f(t)| \le 1$, $t \in \mathbb{R}$.

Since $f \in K_{B_1}$, it follows that the function $g(z) = f(z)/B(z)$, $z \in \mathbb{C}^-$, is in $H^2(\mathbb{C}^-)$. We have $|g(t)| = |f(t)|$, $t \in \mathbb{R}$, and $|c_n| = |g(\overline{z}_n)|/|B'(z_n)|$. Applying Lemma 3.4 to g we get the estimate $\log|g(\overline{z}_n)| \le -C\Omega(n)$, which implies (21).

Now we will show that (22) is fulfilled. By hypotheses, $|f(t)| = o(t^{-N})$, $t \to +\infty$, for any $N > 0$. By Lemma 3.2, f is bounded in the domain Δ defined by (13), and therefore we also have $|f(t)| = o(|t|^{-N})$, $t \to -\infty$, for any $N > 0$. Since f tends to

zero faster than any power, the function $(z+i)^k f(z)$ is in the Hardy class $H^1(\mathbb{C}^+)$ for every $k \geq 0$. Hence,

$$\int_{\mathbb{R}} (t+i)^k \left(\sum_{n=1}^{\infty} \frac{c_n}{t-n+i} \right) dt = 0.$$

Therefore, for $N > k+1$,

$$\lim_{A\to\infty} \int_{\mathbb{R}} \left(\frac{Ai}{Ai-t} \right)^N (t+i)^k \left(\sum_{n=1}^{\infty} \frac{c_n}{t-n+i} \right) dt$$

$$= \lim_{A\to\infty} \sum_{n=1}^{\infty} c_n \int_{\mathbb{R}} \left(\frac{Ai}{Ai-t} \right)^N \frac{(t+i)^k}{t-n+i} \, dt = 0.$$

By the residue calculus (in the lower half-plane), we have

$$\lim_{A\to\infty} \sum_{n=1}^{\infty} \left(\frac{Ai}{Ai-n+i} \right)^N n^k c_n = 0$$

and, finally,

$$\sum_{n=1}^{\infty} n^k c_n = 0. \qquad \bigcirc$$

Proof of Lemma 3.6. Let us define $f \in K_{B_1}$ by formula (19). Then (22) implies for any $k \in \mathbb{N}$

$$(t+i)^k f(t) - \sum_{n\in\mathbb{N}} \frac{c_n n^k}{t-n+i} = 0.$$

Hence,

$$|f(t)| \leq \inf_{k\in\mathbb{Z}_+} \frac{1}{|t|^k} \sum_{n\in\mathbb{N}} |c_n| n^k.$$

Since $|c_n| \leq n^{-3} w(n)$, we have

$$|f(t)| \leq C \inf_{k\in\mathbb{Z}_+} \frac{1}{|t|^k} \sup_{n\in\mathbb{N}} [w(n) n^{k-1}].$$

It is easy to show that for $|t| \geq 1$

$$\inf_{k\in\mathbb{Z}_+} \frac{1}{|t|^k} \sup_{n\in\mathbb{N}} [w(n) n^{k-1}] \leq \inf_{p\geq 0} \frac{1}{|t|^p} \sup_{q\geq 1} [w(q) q^p].$$

Recall that if G is a function on $[0,\infty)$, then its Legendre transform $G^{\#}$ is defined as

$$G^{\#}(x) = \sup_{t\geq 0} (xt - G(t)).$$

Put $G(t) = \Omega(e^t)$. Then

$$\sup_{q\geq 1} w(q)q^p = \sup_{q\geq 1} e^{p\log q - \Omega(q)} = e^{G^{\#}(p)}$$

and

$$\inf_{p\geq 0} e^{G^{\#}(p) - p\log|t|} = e^{-(G^{\#})^{\#}(\log|t|)}.$$

It is well known that for convex functions G we have $(G^{\#})^{\#} = G$ and so

$$|f(t)| \leq e^{-G(\log|t|)} = w(t).$$

The proof is completed. ○

Now we will use the following theorem on analytic quasianalyticity, which is due to P. Koosis [16]. Many results of this type were obtained by M.M. Dzhrbashyan, L. Carleson, B. R.-Salinas and B.I. Korenblum in 1940-s–1960-s. We denote by C_A^∞ the class of functions analytic in the unit disc $\mathbb{D}$ and infinitely differentiable in its closure.

Theorem 3.7. *Let $\{w(n)\}_{n\in\mathbb{N}}$ be a positive sequence such that $w(n) = o(n^{-l})$, $n \to \infty$, for any $l > 0$. Put for $z \in \mathbb{C}$*

$$w_*(z) = \sup\{p(z) :\ p \text{ is a polynomial and } \ |p(n)w(n)| \leq 1, n \in \mathbb{Z}_+\}.$$

Then the following statements are equivalent:

1. There exists a nonzero function $f \in C_A^\infty$, $f(z) = \sum_{n\geq 0} c_n z^n$, such that $|c_n| \leq w(n)$ and $f^{(k)}(1) = 0$, $k \in \mathbb{Z}_+$;

2.

$$\sum_{n=1}^{\infty} n^{-3/2} \log w_*(n) < \infty.$$

We will use only implication 2$\Longrightarrow$1. Note that $w_*(n) \leq 1/w(n)$. Thus, if the integral (5) converges, then there exists a nonzero function f as in Statement 1 above.

Now Theorem 1.1 follows almost immediately from Theorems 3.1 and 3.7.

Proof of Theorem 1.1. Statement 1 is a particular case of Theorem 3.1.

Assume that w is a positive nonincreasing function such that the integral (5) converges. Without loss of generality we may assume that $w(t) \equiv 1$, $t \in [0, 1]$. We replace w by a smaller regular majorant w_1 (see the definition given before Lemma 3.6). Put

$$\Omega_1(x) = \int_0^{ex} \frac{\Omega(t)}{t} dt.$$

Then

$$\Omega_1(x) \ge \int_{|x|}^{e|x|} \frac{\Omega(t)}{t}dt \ge \Omega(|x|)$$

and, consequently, $w_1 = e^{-\Omega_1} \le e^{-\Omega} = w$. Since $\Omega_1'(x) = \Omega(ex)/x$, the function $x\Omega_1'(x)$ is nondecreasing and so w_1 is regular. Finally we put $w_2(t) = (t+1)^{-3}w_1(t)$, $t \ge 0$. It is clear that the convergence of the integral (5) implies that

$$\int_1^\infty t^{-3/2} \log \frac{1}{w_2(t)} dt < \infty.$$

Then, by Theorem 3.7, there exists a nonzero function $f \in C_A^\infty$, $f(z) = \sum_{n=1}^\infty c_n z^n$, such that $|c_n| \le w_2(n)$, $n \in \mathbb{N}$, and $f^{(k)}(1) = 0$, $k \in \mathbb{Z}_+$. The latter condition means that

$$\sum_{n=1}^\infty c_n n^k = 0, \qquad k \in \mathbb{Z}_+.$$

Since w_1 is a regular majorant and $|c_n| \le n^{-3}w_1(n)$, $n \in \mathbb{N}$, it follows from Lemma 3.6 that $w_1 \in \mathrm{Adm}(B_1)$. Hence, $w(|x|) \in \mathrm{Adm}(B_1)$ and, in particular, $w \in \mathrm{Adm}_+(B_1)$. ○

§4. Majorants on the negative semiaxis

Proof of Theorem 1.2. We start with the proof of the sharpness of the exponent 1/2 on the negative semiaxis. Assume that

$$|t|^{1/2} = o(\Omega(t)), \quad t \to -\infty. \tag{23}$$

We will show that $w \notin \mathrm{Adm}_-(B_1)$.

Let Δ be the domain defined by (13). Then, by Lemma 3.2, each function $f \in K_{B_1}$ is analytic and bounded in Δ. Recall that we denote by η the conformal mapping (14) of $\mathbb{C}^+$ onto Δ and we have

$$\eta(z) \sim a_2 z^2/2, \qquad |z| \to \infty, \quad z \in \mathbb{C}^+, \tag{24}$$

where $a_2 > 0$. Hence, $\eta^{-1}(-x) \in \{z : \operatorname{Im} z \ge |\operatorname{Re} z|\}$ for sufficiently large $x > 0$. Also, if we put $\Gamma = \eta^{-1}((-\infty, 0])$, then for $z \in \Gamma$ we have $|\eta(z)| \ge c|z|^2$ for some $c > 0$ when $|z|$ is sufficiently large.

Let $f \in K_{B_1}$ and $|f(t)| \le w(t)$, $t < 0$. Put $g(z) = f(\eta(z))$. Then g is a bounded analytic function in $\mathbb{C}^+$. By (23), for $z \in \Gamma$ we have

$$|g(z)| \le w(|\eta(z)|) \le e^{-C|z|}, \qquad |z| \to \infty, \tag{25}$$

for any $C > 0$. The estimate (24) and an elementary argument using the theorem on two constants permit us to obtain (25) for any $C > 0$ and $z \in i\mathbb{R}_+$. Hence, by the Phragmén–Lindelöf principle, $g \equiv 0$ in $\mathbb{C}^+$.

Now we turn to the proof of admissibility of the majorants $w(t) = \exp(-A|t|^{1/2})$ on the negative semiaxis. In fact, we prove the following stronger result:

Theorem 4.1. *Let B_β be the Blaschke product with the zeros $n^\beta + i$, $n \in \mathbb{N}$, $\beta > 1/2$, and let*

$$W_A(t) = \exp(-A|t|^{1/2}).$$

1. If $A < \pi$, then $W_A \in \mathrm{Adm}_-(B_2)$.

2. If $1 \le \beta < 2$, then the majorant W_A belongs to the class $\mathrm{Adm}_-(B_\beta)$ for any $A > 0$.

Proof of Statement 1. We consider an auxiliary Blaschke product B° with the zeros at the points $-1+i$, i and $(\rho n)^2+i$, $n \in \mathbb{N}$. We will show that $W_A \in \mathrm{Adm}_-(B^\circ)$ for any $A < \pi/\rho$.

Consider the entire function

$$E(z) = \prod_{n\in\mathbb{N}} \left(1 - \frac{z}{(\rho n)^2 - i}\right) = c\,\frac{\sin(\pi\rho^{-1}\sqrt{z+i})}{\sqrt{z+i}} \tag{26}$$

where c is some absolute constant. Then we have

$$\log|E(x)| \sim \frac{\pi|x|^{1/2}}{\rho}, \qquad x \to -\infty. \tag{27}$$

Now let $x \in (k-1/2, k+1/2)$, $k \in \mathbb{N}$. We write

$$|E(\rho^2x^2)| = \left|\frac{\rho^2k^2 - \rho^2x^2 - i}{\rho^2k^2 + i}\right| \prod_{n\ne k}\left|1 - \frac{\rho^2x^2}{\rho^2n^2 - i}\right|.$$

It is easy to see that there exist positive constants C_1 and C_2 depending on ρ but not depending on k and x such that

$$C_1 \le \prod_{n\ne k}\left|1 - \frac{\rho^2x^2}{\rho^2n^2}\right|^{-1}\left|1 - \frac{\rho^2x^2}{\rho^2n^2 - i}\right| \le C_2.$$

Hence

$$|E(\rho^2x^2)| \asymp \left|\frac{\rho^2k^2 - \rho^2x^2 - i}{\rho^2k^2 + i}\right| \cdot \frac{|\sin\pi x|}{\pi|x|}\left|1 - \frac{x^2}{k^2}\right|^{-1},$$

and it follows that

$$\frac{C_3}{k^2} \le |E(\rho^2x^2)| \le \frac{C_4}{k}. \tag{28}$$

Thus,

$$|E(x)| \ge C_5x^{-1}, \qquad x > 1. \tag{29}$$

Now, put $E^\circ(z) = (z+i)(z+1+i)E(z)$. Then $B^\circ = (E^\circ)^*/E^\circ$. It follows from the estimates (27) and (29) that for any $A < \pi/\rho$ there exist positive constants C_6 and C_7 such that

$$|E^\circ(x)|^{-1} \le C_6 \exp(-A|x|^{1/2}), \qquad x < 0,$$

and

$$|E^\circ(x)|^{-1} \le C_7 |x|^{-1}, \qquad x > 1.$$

Hence, $1/E^\circ \in L^2(\mathbb{R})$. It follows immediately from the definition of E° that $|E^\circ(x+iy)|$ is an increasing function of $y \ge 0$ for each $x \in \mathbb{R}$ and so $1/E^\circ \in H^2$. Clearly, in this case $1/E^\circ \in K_{B^\circ}$ and, in particular, $1/|E^\circ| \in \mathrm{Adm}_-(B^\circ)$ (by Theorem 2.6, the function $1/|E^\circ|$ is a positive minimal admissible majorant for K_{B°). In particular, the majorant W_A is in $\mathrm{Adm}_-(B^\circ)$ for any $A < \pi/\rho$.

Now let $\rho > 1$. To prove Statement 1 it suffices to show that $\mathrm{Adm}_-(B^\circ) \subset \mathrm{Adm}_-(B_2)$. Let φ be an increasing branch of the argument of the Blaschke product B_2 and let ψ be an increasing branch of the argument of B°. Then

$$\varphi'(t) = 2\sum_{n\in\mathbb{N}} \frac{1}{(t-n^2)^2+1},$$

$$\psi'(t) = \frac{2}{t^2+1} + \frac{2}{(t+1)^2+1} + 2\sum_{n\in\mathbb{N}} \frac{1}{(t-\rho n^2)^2+1}.$$

For $n \in \mathbb{N}$ and $M > 0$ put $d_n = (Mn)^2$. Since $\rho > 1$, for sufficiently large M we have

$$\int_{d_n}^{d_{n+1}} \varphi'(t)dt \asymp 1 \qquad \text{and} \qquad \int_{d_n}^{d_{n+1}} (\varphi'(t) - \psi'(t))dt \asymp 1.$$

Note also that there is $C > 0$ such that the function $\varphi - \psi$ is an increasing Lipschitz function for $t < -C$. Hence, $\varphi - \psi$ is mainly increasing and, by the remark after Corollary 2.4, $\mathrm{Adm}_-(B^\circ) \subset \mathrm{Adm}_-(B_2)$. Thus, the majorant W_A is in $\mathrm{Adm}_-(B_2)$ for any $A < \pi/\rho$. Since ρ is an arbitrary number greater than 1, the proof of Statement 1 is completed.

Proof of Statement 2. Let B° be the same Blaschke product with the zeros $-1+i$, i and $(\rho n)^2 + i$, $n \in \mathbb{N}$. But this time we will assume ρ to be small. We have shown in the proof of Statement 1 that W_A belongs to $\mathrm{Adm}_-(B^\circ)$ for any $A < \pi/\rho$.

Recall that we denote by φ_β an increasing continuous branch of the argument of B_β (see formula (16)). If $1 \le \beta < 2$, then it is easy to see that $\varphi_\beta - \psi$ is a mainly increasing function for any $\rho > 0$ (take $d_n = (Mn)^\beta$, $n \in \mathbb{N}$, for a sufficiently large M). Now, by Corollary 2.4, $\mathrm{Adm}(B^\circ) \subset \mathrm{Adm}(B_\beta)$ for any $\rho > 0$ and therefore $W_A \in \mathrm{Adm}_-(B_\beta)$ for any $A > 0$. ◯

Remark. It is easy to see that the constant π in Statement 1 is sharp, that is,

$W_A \notin \mathrm{Adm}_-(B_2)$ for $A \ge \pi$. Indeed, let E be the function defined in (26) with $\rho = 1$. Then $|E(x)| \asymp |x|^{-1/2} \exp(\pi |x|^{1/2})$, $x < -1$, and, by (28), $|E(x)| \le Cx^{-1/2}$, $x > 1$. Assume that $f \in K_{B_2}$ and $|f(t)| \le W_A(t)$, $t < 0$, where $A > \pi$. Note that, by Lemma 3.2, f is bounded on $\mathbb{R}$. It is easily seen that $F = fE$ is an entire function of order at most $1/2$ (see [10, Theorem 3.1]), which is bounded on the real axis and $|F(t)| \to 0$, $t \to -\infty$. Hence, $F \equiv 0$.

The following statement shows that the fast decay on the negative semiaxis is compatible with any admissible decay on the positive semiaxis from Theorem 3.1.

Corollary 4.2. *For any $A > 0$ and for any nonincreasing majorant w with the finite integral (5) there exists a nonzero function $f \in K_{B_1}$ such that*

$$|f(t)| \le w(t), \quad t > 0, \quad \text{and} \quad |f(t)| \le W_A(t), \quad t < 0.$$

Proof. Let $\widetilde{B}$ be the Blaschke product with zeros $\tau n + i$, $n \in \mathbb{N}$, where $\tau > 1$. Clearly, Theorem 1.1 is applicable also to the space $K_{\widetilde{B}}$ and, in particular, each nonincreasing majorant w such that the integral (5) converges is in the class $\mathrm{Adm}_+(\widetilde{B})$.

Now let $D = B_{3/2}\widetilde{B}$, where $B_{3/2}$ is the Blaschke product with the zeros $n^{3/2} + i$, $n \in \mathbb{N}$. Note that, by Lemma 3.2, $K_{B_{3/2}} \subset L^\infty(\mathbb{R})$ and $K_{\widetilde{B}} \subset L^\infty(\mathbb{R})$. Therefore, if $f \in K_{B_{3/2}}$ and $g \in K_{\widetilde{B}}$, then $fg \in K_D$.

By Statement 2 of Theorem 4.1, there exists $f \in K_{B_{3/2}}$ such that $|f(t)| \le W_A(t)$, $t < 0$, and there exists $g \in K_{\widetilde{B}}$ such that $|g(t)| \le w(t)$, $t > 0$. Hence the majorant

$$W(t) = \begin{cases} W_A(t), & t < 0, \\ w(t), & t > 0. \end{cases}$$

is admissible for K_D, since $|f(t)g(t)| \le CW(t)$, $t \in \mathbb{R}$.

To complete the proof note that $\arg B_1 - \arg D$ is a mainly increasing function since $\tau > 1$. Now we apply Corollary 2.4 and see that $W \in \mathrm{Adm}(B_1)$. ◯

We conclude this section with the proof of Theorem 1.3.

Proof of Theorem 1.3. For $h, \rho > 0$ let $B_{h,\rho}$ be the Blaschke product with the zeros $n/\rho + ih$, $n \in \mathbb{N}$. It is clear that all results of Theorems 1.1 and 1.2 are true also for the products $B_{h,\rho}$. Denote by $\varphi_{h,\rho}$ an increasing branch of the argument of $B_{h,\rho}$.

Let all z_n lie in a half-strip $[0, \infty) \times [\delta, M]$ and satisfy the condition (6). Then there exist $L > 0$ and $N \in \mathbb{N}$ such that each rectangle $[x, x + L] \times [\delta, M]$ contains at least one and no more than N of the zeros z_n. It is easy to see that $\varphi'(x) \asymp 1$, $x > 0$ [11, Theorem 3.4], and $\varphi'(x) \asymp |x|^{-1}$, $x < -1$, where φ is an increasing branch of the argument of B. Then there exist positive numbers h_1, ρ_1, h_2, and ρ_2 such that

$$\varphi'_{h_1,\rho_1}(x) < \varphi'(x) < \varphi'_{h_2,\rho_2}(x), \qquad x \in \mathbb{R},$$

and, thus, the functions $\varphi - \varphi_{h_1,\rho_1}$ and $\varphi_{h_2,\rho_2} - \varphi$ are increasing. By Corollary 2.4,

$$\mathrm{Adm}(B_{h_1,\rho_1}) \subset \mathrm{Adm}(B) \subset \mathrm{Adm}(B_{h_2,\rho_2}),$$

$$\mathrm{Adm}_+(B_{h_1,\rho_1}) \subset \mathrm{Adm}_+(B) \subset \mathrm{Adm}_+(B_{h_2,\rho_2}),$$

and

$$\mathrm{Adm}_-(B_{h_1,\rho_1}) \subset \mathrm{Adm}_-(B) \subset \mathrm{Adm}_-(B_{h_2,\rho_2}). \qquad \bigcirc$$

§5. Power growth of zeros

The proof of Theorem 1.4 consists of a few steps. First we obtain the formulas (7) and (8) for the case $\beta \geq 1$. Then, making use of a method of [11] we complete the proof of the formula (7). Finally, we will show that $\alpha_-(\beta) = 1$ for $\beta < 1$.

We will use repeatedly the following lemma.

Lemma 5.1. *Let $\beta > \gamma > 1/2$. Then* $\mathrm{Adm}(B_\beta) \subset \mathrm{Adm}(B_\gamma)$. *In particular, the functions α, α_+ and α_- are nonincreasing functions of $\beta > 1/2$.*

Proof. Denote by φ_β an increasing branch of the argument of B_β. Then

$$\varphi_\beta'(t) = 2\sum_{n\in\mathbb{N}} \frac{1}{(t-n^\beta)^2+1}.$$

It is easy to see that for $\beta \geq 1$ we have

$$\varphi_\beta'(t) \asymp \frac{1}{(t-(n_\beta(t))^\beta)^2+1}, \qquad t > 0, \tag{30}$$

where $n_\beta(t)$ is the integer closest to $t^{1/\beta}$. On the other hand, for $1/2 < \beta < 1$ we have shown in the proof of Lemma 3.2 that

$$\varphi_\beta'(t) \asymp t^{-1+1/\beta}, \qquad t > 1. \tag{31}$$

Finally, for $\beta > 1/2$,

$$\varphi_\beta'(t) \asymp |t|^{-2+1/\beta}, \qquad t < -1. \tag{32}$$

Let $\beta > \gamma$. For $M > 0$ and $n \in \mathbb{N}$ put $d_n = (Mn)^\gamma$. It follows from (30) and (31) that for sufficiently large M we have

$$\int_{d_n}^{d_{n+1}} \varphi_\gamma'(t)dt \asymp 1 \qquad \text{and} \qquad \int_{d_n}^{d_{n+1}} (\varphi_\gamma'(t) - \varphi_\beta'(t))dt \asymp 1.$$

It is also clear that the function $\varphi_\gamma - \varphi_\beta$ is increasing when $t < 0$ and $|t|$ is sufficiently large. Thus, the function $\varphi_\gamma - \varphi_\beta$ is mainly increasing. By Corollary 2.4, $\mathrm{Adm}(B_\beta) \subset$

$\mathrm{Adm}(B_\gamma)$ and, consequently, $\alpha(\beta) \le \alpha(\gamma)$. Analogously, by the remark following Corollary 2.4, $\alpha_+(\beta)$ and $\alpha_-(\beta)$ are nonincreasing functions of β. ◯

We will also need the following lemma on asymptotics of certain canonical products.

Lemma 5.2. *For $\beta > 2$ consider the entire function*

$$E_\beta(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^\beta - i}\right). \tag{33}$$

Then

$$\log |E_\beta(x)| \asymp |x|^{1/\beta}, \qquad |x| \to \infty.$$

Proof. The asymptotics of E_β outside an exceptional set is given in [18, Chapter II, Theorem 5]. For $x \to -\infty$ we have $\log |E_\beta(x)| \asymp |x|^{1/\beta}$. For sufficiently small $\varepsilon > 0$, $\varepsilon_1 > 0$ we have the estimate

$$\log |E_\beta(z)| \asymp |z|^{1/\beta}, \qquad |z| \to \infty, \quad |\arg z| < \varepsilon, \quad z \notin \cup_n D_n,$$

where $D_n = \{w \in \mathbb{C} : |w - (n^\beta - i)| < \varepsilon_1 n^{\beta-1}\}$. Dividing E_β by $(z - (n^\beta - i))$ and applying the maximum principle in the discs D_n we conclude that $\log |E_\beta(x)| \asymp x^{1/\beta}$, $x \to \infty$. ◯

Proof of (7) and (8) for the case $\beta \ge 1$. First, assume that $\beta > 2$. Clearly, $B_\beta = E_\beta^*/E_\beta$, where the entire function E_β is defined by (33). By Lemma 5.2, $\log |E_\beta(x)| \asymp |x|^{1/\beta}$, $|x| \to \infty$. Hence, $1 \in \mathcal{H}(E_\beta)$ and, by Theorem 2.6, $1/|E_\beta|$ is the minimal admissible majorant for K_{B_β}. Thus, $\alpha(\beta) = 1/\beta$.

Also we have $\alpha_+(\beta) \ge 1/\beta$ and $\alpha_-(\beta) \ge 1/\beta$. To prove the converse inequalities let us show that w_γ does not belong to $\mathrm{Adm}_+(B_\beta)$ or $\mathrm{Adm}_-(B_\beta)$ when $\gamma > 1/\beta$. Indeed, if a function $f \in K_{B_\beta}$ satisfies $|f(t)| \le e^{-t^\gamma}$, $t > 0$, then $F = fE_\beta$ is an entire function of order at most $1/\beta < 1/2$ and $|F(t)| \to 0$, $t \to +\infty$. Therefore, by the Phragmén–Lindelöf principle, $F \equiv 0$. The same argument works for $\alpha_-(\beta)$.

So, we have shown that for $\beta > 2$

$$\alpha(\beta) = \alpha_+(\beta) = \alpha_-(\beta) = 1/\beta.$$

By Lemma 5.1, the functions α, α_+ and α_- are nonincreasing. Since, by Theorem 1.1, $\alpha(1) = 1/2$ and $\lim_{\beta \to 2+0} \alpha(\beta) = 1/2$, we see that $\alpha(\beta) = 1/2$ for $1 < \beta \le 2$. In a similar way we get $\alpha_+(\beta) = \alpha_-(\beta) = 1/2$ for $1 < \beta \le 2$. ◯

Now we obtain an estimate from below for $\alpha(\beta)$ in the case $\beta < 1$.

Lemma 5.3. *Let $1/2 < \beta < 1$. Let w be an even function nonincreasing on $[0, \infty)$ and satisfying the condition*

$$\Omega(t) = o(t^{-1+1/\beta}), \qquad t \to \infty.$$

Then $w \in \mathrm{Adm}(B_\beta)$.

Proof. We apply a method of the paper [11], which was used there to deduce the admissibility of even majorants with convergent logarithmic integral from Corollary 2.3 in the case $\varphi' \asymp 1$.

Without loss of generality let $\Omega(t) = 0$, $|t| \le 1$. We regularize the majorant w by considering the majorants

$$\Omega_1(x) = \int_0^{e|x|} \frac{\Omega(t)}{t} dt$$

and

$$\Omega_2(x) = \int_0^{e|x|} \frac{\Omega_1(t)}{t} dt.$$

Clearly, Ω_1 and Ω_2 are nondecreasing as well as Ω and $\Omega_2(x) \ge \Omega_1(x) \ge \Omega(x)$. We have also $\Omega_2(x) = o(x^{-1+1/\beta})$, $x \to \infty$, and, consequently, $\int_{\mathbb{R}} \Omega_2(x)(1+x^2)^{-1}dx < \infty$. It is shown in [11], Lemma 4.7 that $\widetilde{\Omega}_2$ is a smooth function and its derivative is given by

$$\frac{d\widetilde{\Omega}_2(x)}{dx} = -\frac{1}{\pi} \int_0^\infty \log\left|\frac{1+t}{1-t}\right| \frac{\Omega(e^2xt)}{|x|t} dt, \qquad x \ne 0.$$

Since $\Omega(t) = o(t^{-1+1/\beta})$, $t \to \infty$, by the dominated convergence theorem, we have

$$|x|^{2-1/\beta} \frac{d\widetilde{\Omega}_2(x)}{dx} = -\frac{1}{\pi} \int_0^\infty \frac{1}{t^{2-1/\beta}} \log\left|\frac{1+t}{1-t}\right| \cdot \frac{\Omega(e^2xt)}{(|x|t)^{-1+1/\beta}} dt \to 0$$

when $|x| \to \infty$. Thus, $(\widetilde{\Omega}_2)'(x) = o(|x|^{-2+1/\beta})$, $|x| \to \infty$.

To apply Corollary 2.3 we have to compare the growth of $\varphi_\beta = \arg B_\beta$ with the growth of $\widetilde{\Omega}_2$. It follows from the estimates (31)–(32) that $(\widetilde{\Omega}_2)'(x) = o(\varphi'_\beta(x))$ for sufficiently large $|x|$. By Corollary 2.3, $w_2 = e^{-\Omega_2} \in \mathrm{Adm}(B_\beta)$ and, consequently, $w \in \mathrm{Adm}(B_\beta)$. ◯

Corollary 5.4. $\alpha(\beta) \ge \max(1/2, -1 + 1/\beta)$ *for* $1/2 < \beta < 1$.

Proof. By Lemma 5.3, $w_\gamma \in \mathrm{Adm}(B_\beta)$ for $\gamma < -1 + 1/\beta$. On the other hand, $\alpha(\beta) \ge \alpha(1) = 1/2$. ◯

Remark. Note that the method of Lemma 5.3 does not allow to prove Theorem 1.1. Indeed, in the case $\beta = 1$ we have $\varphi'_1(x) \asymp |x|^{-1}$, $x < -1$, whereas for the majorant $w = w_\gamma$ with $\gamma \in (0, 1/2)$

$$(\widetilde{\Omega}_2)'(x) \le -C|x|^{\gamma-1}, \qquad x < -1,$$

for some $C > 0$. Therefore the function $\varphi + 2\widetilde{\Omega}_2$ is not increasing. Thus, for the case of sparse zeros the sufficient condition of Corollary 2.3 is far from being necessary.

Proof of (7) for the case $1/2 < \beta < 1$**.** By Corollary 5.4, $\alpha(\beta) \ge -1 + 1/\beta$ for $\beta < 1$. Also we have $\alpha(\beta) \ge 1/2$, $\beta < 1$, since the function $\alpha(\beta)$ is nonincreasing. Note that $-1 + 1/\beta = 1/2$ for $\beta = 2/3$.

Let $1/2 < \beta < 2/3$. We show that in this case $\alpha_+(\beta) \le -1+1/\beta$ and, since $\alpha_+(\beta)$ is a nonincreasing function of β, the proof of the formula (7) will be completed.

Assume that $\gamma > -1 + 1/\beta$ and there is a nonzero function f in K_{B_β} such that $|f(t)| \le e^{-t^\gamma}$, $t > 0$. Let us show that f is bounded in the domain Δ defined by (13).

By (32), B'_β is bounded on $(-\infty, 0]$ and we have $f \in H^\infty(\mathbb{C}^+)$ by (11). Applying Lemma 3.4 to the function $g(z) = f(z)/B_\beta(z)$ in the lower half-plane $\mathbb{C}^-$ ($g \in H^2(\mathbb{C}^-)$) we see that

$$\log \left| \frac{f(z)}{B_\beta(z)} \right| \le -C|z|^\gamma, \qquad -\pi/2 \le \arg z < 0,$$

where $\arg z$ stands for the main branch of the argument. Now it follows from (17) that

$$|f(z)| \le C_0, \qquad z \in \Delta \cap \{-\pi/2 \le \arg z \le 0\}.$$

By Lemma 3.2, $\log|f(z)| \le C_1 + C_2|z|^{-1+1/\beta}$, $z \in \Delta$. Applying the Phragmén–Lindelöf principle to the function f in the angle $\{-\pi < \arg z < -\pi/2\}$ we see that f is bounded in Δ. Hence, the function $g = f \circ \eta$, where η is the conformal mapping (14) of $\mathbb{C}^+$ onto Δ, is bounded in $\mathbb{C}^+$. On the other hand, $\eta(t) \asymp t^2$, $t \to \infty$, and, consequently, $\log|g(t)| \le -C_3 t^{2\gamma}$, $t > 1$. Note that $2\gamma > 1$ since $\gamma > -1 + \frac{1}{\beta}$ and $\beta < 2/3$. Hence, $g \equiv 0$ and we got a contradiction. $\bigcirc$

Remark. It is interesting to compare the formula for $\alpha_+(\beta)$ with the results of A. Borichev and M. Sodin on weighted polynomial approximation on discrete subsets of the line ([6], Appendix 2). Let $x_n = n^\beta$, $n \in \mathbb{N}$, and let $w_{\gamma,A}(t) = \exp(-A|t|^\gamma)$, where $A, \gamma > 0$. Consider the space

$$\ell^2(w_{\gamma,A}) = \{f : \{x_n\} \to \mathbb{C} : \sum_{n=1}^{\infty} |f(x_n)|^2 w_{\gamma,A}(x_n) < \infty\}.$$

The following theorem answers the question about the density of the polynomials in the spaces $\ell^2(w_{\gamma,A})$.

If $\beta > 2$, then the polynomials are dense in the space $\ell^2(w_{\gamma,A})$ for $\gamma > 1/\beta$ and are not dense for $\gamma < 1/\beta$; if $\gamma = 1/\beta$, then the polynomials are dense if and only if $A \ge 2\pi \cot \frac{\pi}{\beta}$. If $\beta \le 2$, then the polynomials are dense in $\ell^2(w_{\gamma,A})$ if and only if $\gamma \ge 1/2$.

Thus, for $\beta \ge 2/3$ (but not for $\beta < 2/3$) the limit exponent $\alpha_+(\beta)$ coincides with the limit γ in the theorem of Borichev and Sodin. Moreover, one can deduce the formula for $\alpha_+(\beta)$ from this theorem making use of the representations of the form (18) with rapidly decreasing $|c_n|$ and an argument analogous to Lemma 3.5. Here we prefer to use a more direct approach.

Proof of (8) for the case $1/2 < \beta < 1$. We use once more the smoothing technique of Lemma 5.3. Let $0 < \alpha < 1$. Consider the functions

$$U(t) = \begin{cases} 0, & |t| \le 1, \\ |t|^\alpha - 1, & |t| > 1, \end{cases} \tag{34}$$

and

$$V(t) = \begin{cases} 0, & t \le 1, \\ 1 - t^\alpha, & t > 1. \end{cases}$$

Now let

$$U_1(x) = \int_0^{|x|} \frac{U(t)}{t} dt, \qquad U_2(x) = \int_0^{|x|} \frac{U_1(t)}{t} dt. \tag{35}$$

Analogously, for $x > 0$ let

$$V_1(x) = \int_0^{x} \frac{V(t)}{t} dt, \qquad V_2(x) = \int_0^{x} \frac{V_1(t)}{t} dt.$$

For $x < 0$ let $V_1(x) = V_2(x) = 0$.

We will show that there exist positive constants K and M such that the majorant

$$w = \exp(-KU_2 - MV_2)$$

is in $\mathrm{Adm}_-(B_\beta)$ for any $\beta \in (1/2, 1)$. Clearly, $KU_2(t) + MV_2(t) \asymp |t|^\alpha$, $t < -1$. Thus we obtain the estimate $\alpha_-(\beta) \ge \alpha$. Since α is an arbitrary number from the interval $(0, 1)$, we have $\alpha_-(\beta) = 1$ and our statement will be proved.

By Lemma 4.7 of [11],

$$\frac{d\widetilde{U}_2(x)}{dx} = -\frac{1}{\pi} \int_0^\infty \log\left|\frac{1+t}{1-t}\right| \frac{U(xt)}{|x|t} dt, \qquad x \ne 0.$$

Hence,

$$\frac{d\widetilde{U}_2(x)}{dx} = -\left(\frac{1}{\pi} \int_0^\infty \log\left|\frac{1+t}{1-t}\right| t^{\alpha-1} dt\right) |x|^{\alpha-1} + O\left(\frac{1}{|x|}\right), \qquad |x| > 1. \tag{36}$$

Analogously, it is easy to show that

$$\frac{d\widetilde{V}_2(x)}{dx} = -\frac{1}{\pi} \int_0^\infty \log\left(\frac{1+t}{t}\right) \frac{V(|x|t)}{|x|t} dt, \qquad x < 0,$$

and

$$\frac{d\widetilde{V}_2(x)}{dx} = \frac{1}{\pi}\int\limits_0^\infty \log\left|\frac{1-t}{t}\right| \frac{V(xt)}{xt}dt, \qquad x > 0.$$

Hence, for $x < -2$ we have

$$\frac{d\widetilde{V}_2(x)}{dx} = \left(\frac{1}{\pi}\int\limits_0^\infty \log\left(\frac{1+t}{t}\right) t^{\alpha-1}dt\right)|x|^{\alpha-1} - \left(\frac{1}{\pi}\int\limits_0^{1/|x|} \log\left(\frac{1+t}{t}\right) t^{\alpha-1}dt\right)|x|^{\alpha-1}$$

$$-\frac{1}{\pi|x|}\int\limits_{1/|x|}^\infty \log\left(\frac{1+t}{t}\right)\frac{dt}{t} = \left(\frac{1}{\pi}\int\limits_0^\infty \log\left(\frac{1+t}{t}\right) t^{\alpha-1}dt\right)|x|^{\alpha-1} + O\left(\frac{\log^2|x|}{|x|}\right).$$

Finally, it is easy to see that

$$\frac{d\widetilde{V}_2(x)}{dx} = O(x^{\alpha-1}), \qquad x > 2.$$

Now we take two positive constants K and M such that

$$K\int\limits_0^\infty \log\left|\frac{1+t}{1-t}\right| t^{\alpha-1}dt = M\int\limits_0^\infty \log\left(\frac{1+t}{t}\right) t^{\alpha-1}dt.$$

Hence, the function $\Omega = KU_2 + MV_2$ satisfies the following asymptotic equalities:

$$\frac{d\widetilde{\Omega}(x)}{dx} = O\left(\frac{\log^2|x|}{|x|}\right), \qquad x < -2;$$

$$\frac{d\widetilde{\Omega}(x)}{dx} = O(x^{\alpha-1}), \qquad x > 2.$$

To apply Corollary 2.3 we should compare the growth of $\widetilde{\Omega}$ with the growth of the argument φ_β of the product B_β. By (31) and (32), using that $\beta < 1$, we obtain

$$(\widetilde{\Omega})'(x) = o(\varphi_\beta'(x)), \qquad |x| \to \infty.$$

Moreover, $\varphi_\beta + 2\widetilde{\Omega}$ is a mainly increasing function. Indeed, $\varphi_\beta + 2\widetilde{\Omega}$ is Lipschitz and increasing on $(-\infty, -R)$ for some $R > 0$. To show that $\varphi_\beta + 2\widetilde{\Omega}$ is mainly increasing on $(0, \infty)$, put $d_n = (Mn)^\beta$ for a sufficiently large M. Hence, $w = e^{-\Omega}$ is admissible, which completes the proof of the theorem. ◯

Remark 5.5. It should be noted that nonzero functions in K_β, $1/2 < \beta < 1$, with fast decrease at $-\infty$ should be necessarily unbounded on $[0, \infty)$. For example, any nonzero function $f \in K_{B_\beta}$, $2/3 < \beta < 1$, which is majorized on $(-\infty, 0]$ by w_α with $\alpha \in (1/2, 1)$, is unbounded on $[0, \infty)$.

Indeed, assume that $|f(t)| \le w_\alpha(t)$, $t<0$, and $|f(t)| \le 1$, $t>0$. Put $g = f\circ\eta$ where η is the conformal mapping (14). Then, by Lemmas 3.2, 3.3 and by the arguments analogous to those in the proof of Theorem 1.2, we have $|g(t)| \le C_1$, $t>0$, $\log|g(z)| \le C_2|z|^{-2+2/\beta}$, $z\in\mathbb{C}^+$, and $\log|g(iy)| \le -C_3 y^{2\alpha}$, $y>0$. Note that $-2+2/\beta < 1$, since $2/3<\beta<1$, and $2\alpha>1$. Now $g\equiv 0$ by a variant of the Phragmén–Lindelöf principle.

We conclude this section with the formula for the fastest possible decay of elements of K_B for two-sided sequences with power growth. Let B be the Blaschke product with the zeros

$$z_n = \begin{cases} n^\beta + i, & n\in\mathbb{Z},\ n>0, \\ -|n|^\gamma + i, & n\in\mathbb{Z},\ n<0. \end{cases}$$

where $\beta,\gamma > 1/2$, and let

$$\alpha(\beta,\gamma) = \sup\{\alpha : w_\alpha \in \operatorname{Adm}(B)\}.$$

Theorem 5.6. *Let $\beta\ge\gamma>1/2$. If $\beta\le 1$, then $\alpha(\beta,\gamma)=1$. If $\beta>1$, then*

$$\alpha(\beta,\gamma) = \max\left(\frac{1}{\beta}, \alpha(\gamma)\right).$$

Proof. The case $\beta\le 1$ is obvious. Let $\beta>1$ and let $\rho > \max(\frac{1}{\beta},\alpha(\gamma))$. Assume that $f\in K_B$ and $|f(t)| \le e^{-|t|^\rho}$, $t\in\mathbb{R}$. By Lemma 3.4, $|f(z)| \le e^{-C|z|^\rho}$, $z\in\mathbb{C}^+$.

Consider the function E_β defined as in (33) and put $g = fE_\beta$. It is clear that the function E_β is of order at most $1/\beta$. Hence,

$$|g(z)| \le e^{-C_1|z|^\rho}, \qquad z\in\mathbb{C}^+,$$

for some $C_1>0$. Thus, $g\in H^2$ and, since g is meromorphic in $\mathbb{C}$ and all its poles are in the set $-|n|^\gamma - i$, $n<0$, we conclude that $g\in K_{B^\#_\gamma}$, where $B^\#_\gamma$ is the Blaschke product with the zeros $-|n|^\gamma + i$, $n<0$.

By our assumption $\rho>\alpha(\gamma)$ and $|g(t)| \le e^{-C_1|t|^\rho}$, $t\in\mathbb{R}$. It follows that $g\equiv 0$. So we see that $w_\rho \notin \operatorname{Adm}(B)$ when $\rho > \max(\frac{1}{\beta},\alpha(\gamma))$. Hence, $\alpha(\beta,\gamma) \le \max(\frac{1}{\beta},\alpha(\gamma))$

Since $K_{B^\#_\gamma} \subset K_B$ it follows that $\alpha(\beta,\gamma)\ge\alpha(\gamma)$. We show that $\alpha(\beta,\gamma)\ge 1/\beta$, $\beta>1$. Let $\alpha(\gamma) < 1/\beta$ and $\rho<1/\beta$, and let U be the function (34) with $\alpha=\rho$. Applying the smoothing procedure (35), we get the majorant $e^{-U_2} \le e w_\rho$, and, by (36), $(\widetilde{U}_2)'(x) \asymp |x|^{\rho-1}$, $|x|>1$.

Denote by φ an increasing branch of the argument of B. Clearly,

$$\varphi'(x) = \varphi'_\beta(x) + \varphi'_\gamma(-x), \quad x\in\mathbb{R}.$$

Let us show that the function $\varphi+2\widetilde{U}_2$ is mainly increasing. Let $d_n = n^\beta$, $n \in \mathbb{N}$. Since $\alpha(\gamma) < 1/\beta$, we have $1/\gamma < 1+1/\beta$. It follows from (30)–(32) that $\varphi(d_{n+1})-\varphi(d_n) \asymp 1$, whereas for $\rho < 1/\beta$

$$\sup_{x_1,\, x_2\in[d_n,d_{n+1}]} |\widetilde{U}_2(x_1) - \widetilde{U}_2(x_2)| \le C((n+1)^{\rho\beta} - n^{\rho\beta}) \to 0, \quad n \to \infty.$$

Thus, we have a sequence $\{d_n\}$ satisfying the conditions of Theorem 2.2. On the negative semiaxis φ grows even faster than for $x > 0$ since $\gamma \le \beta$. Hence, for $x < 0$ the conditions of Theorem 2.2 are also satisfied (one can take $d_n = -|n|^\gamma$, $n < 0$) and so the function $\varphi + \widetilde{U}_2$ is mainly increasing. By Theorem 2.2, $e^{-U_2} \in \mathrm{Adm}(B)$ and, consequently, the majorant w_ρ is admissible for K_B whenever $\rho < 1/\beta$. Therefore $\alpha(\beta,\gamma) \ge 1/\beta$, which completes the proof. ◯

Remark. The case $\beta = \gamma < 1$ is considered in more detail in [11] where certian conditions sufficient for admissibility are given, in terms of $\widetilde{\Omega}$. Some admissibility criteria for two-sided sequences of zeros z_n, $n \in \mathbb{Z}$, are also obtained in [4] where the results are stated in terms of Ω (not $\widetilde{\Omega}$) and some oscillating Ω are studied.

§6. Tangential zeros

In this section we prove Theorem 1.5. Recall that now B is the Blaschke product with zeros $z_n = n + iy_n$, $n \in \mathbb{Z}$, where $0 < y_n \le 1$, the sequence y_n is even and nonincreasing for $n \ge 0$.

In what follows we will need the functions $y(t)$, $Y(t)$, $t \in \mathbb{R}$, defined in Theorem 1.5. Clearly, the integral

$$\mathcal{L}(y) = \int_{\mathbb{R}} \frac{Y(t)}{1+t^2} dt$$

converges if and only if the series (9) converges.

Proof of Statement 1 of Theorem 1.5. Let majorant w be even, nonincreasing on $\mathbb{R}_+$, and let $\mathcal{L}(w) < \infty$ (without loss of generality we assume that $w \le 1$). We will show that $w \in \mathrm{Adm}(B)$ under condition (9).

Let us consider the Blaschke product $B^\#$ with the zeros $\zeta_n = n + i(y_n + 1)$. It is clear that $(\arg B^\#)' \asymp 1$ (by the $\arg B^\#$ we mean an increasing branch). Hence, by Corollary 2.5, $\mathrm{Adm}(B^\#) \supset \mathrm{Adm}(e^{ibz})$ for some $b > 0$. In particular, the majorant

$$w_1(t) = y^2(t) w^{2A}(t+1)(1+t^2)^{-1} \tag{37}$$

is in $\mathrm{Adm}(B^\#)$ for any $A > 1$, since $\mathcal{L}(w_1) < \infty$. Thus, there exists a nonzero function $g \in K_{B^\#}$ such that

$$|g(t)| \le y^2(t) w^{2A}(t+1)(1+t^2)^{-1}, \qquad t \in \mathbb{R}.$$

The sequence $\{\zeta_n\}$ is interpolating. Therefore g may be represented as

$$g(z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z - n + i(y_n + 1)}.$$

Moreover, by Lemma 3.4 and a form of Lemma 3.5, we have

$$|c_n| \le C_1 y_n w^A(n+1)(n^2+1)^{-1/2}.$$

Put $d_n = c_n/\sqrt{y_n}$ and consider the function

$$f(z) = g(z-i) = \sum_{n\in\mathbb{Z}} \frac{\sqrt{y_n} d_n}{z - n + iy_n}.$$

Since the sequence $z_n = n + iy_n$ is interpolating and $\{d_n\} \in \ell^2$, the function f belongs to K_B. It remains to verify that $|f(t)| \le Cw(t)$, $t \in \mathbb{R}$. Note first that

$$\left|\frac{c_n}{t - n + iy_n}\right| \le C_1 w^A(n+1), \qquad t \in \mathbb{R}.$$

After that, we note that the function

$$g_n(z) = g(z) - \frac{c_n}{z - n + iy_n}$$

is analytic in $\Omega_n = \{z : n - 2/3 \le \operatorname{Re} z \le n + 2/3, -1 \le \operatorname{Im} z \le 1\}$ and, by (11)–(12), we have $|g_n(z)| \le C_2$, $z \in \Omega_n$. At the same time,

$$|g_n(t+i)| \le w^A(t+1) + C_1 w^A(n+1), \qquad t \in [n - 2/3, n + 2/3].$$

Therefore, by the theorem on two constants, $|g_n(t)| \le C_3 w(t)$, $t \in [n - 1/2, n + 1/2]$, for sufficiently large constant A. ◯

Proof of Statement 2 of Theorem 1.5. Now we have an additional assumption that the function $Y(e^x)$ is a convex function of x.

Assume that the majorant w decays faster than any power, that is, $w(t) = o(|t|^{-N})$, $|t| \to \infty$, for any $N > 0$. Let f be a function in K_B such that $|f| \le w$ on $\mathbb{R}$. Since the sequence $\{z_n\}$ is interpolating we have the representation

$$f(z) = \sum_{n\in\mathbb{Z}} \frac{\sqrt{y_n} c_n}{z - n + iy_n},$$

where $\{c_n\} \in \ell^2$. Proceeding as in the proof of Lemma 3.5, we obtain the equalities

$$\sum_{n\in\mathbb{Z}} \sqrt{y_n} c_n (n - iy_n)^k = 0, \qquad k \in \mathbb{Z}_+. \tag{38}$$

Consider the function $F(x) = \int_{\mathbb{R}} f(t)e^{-itx}dt$, $x \in \mathbb{R}$. Then

$$F(x) = G(x) = -2\pi i \sum_{n\in\mathbb{Z}} \sqrt{y_n} c_n \exp[-ix(n - iy_n)], \qquad x > 0. \tag{39}$$

Since $y_n \le 1$ it follows from the divergence of (9) and from the convexity of $Y(e^x)$ that the function G defined by (39) is C^∞ smooth on $\mathbb{R}$ and

$$|G^{(k)}(x)| \le C \sup_{n\in\mathbb{N}} \sqrt{y_n}\, n^k e^k, \qquad x \in \mathbb{R}.$$

It follows from (38) that $G^{(k)}(0) = 0$, $k \ge 0$. Furthermore,

$$A_k = \sup_{n\in\mathbb{N}} \sqrt{y_n}\,(en)^k \le \exp\left(\sup_{r>0}[k \log r - \frac{1}{2}Y(r/e)]\right).$$

Put $T(r) = \sup_{k\in\mathbb{Z}_+} r^k/A_k$. Then, by a Legendre transform argument analogous to that in the proof of Lemma 3.6, we get

$$\log T(r) \ge \frac{1}{2}Y(r/e) - C\log r, \qquad r > 1,$$

for some constant C. Therefore $\int_1^\infty r^{-2}\log T(r)dr = \infty$, and the classical Denjoy–Carleman quasianalyticity theorem implies $G \equiv 0$. Since $f \in H^1$ we have

$$\int_{\mathbb{R}} f(t)e^{-itx}dt = 0, \qquad x \le 0.$$

We conclude that $F \equiv 0$ and, hence, $f \equiv 0$. ○

Results analogous to Theorem 1.5 may be obtained for the Blaschke products with power growth of zeros or with one-sided zeros. Let us state the corresponding result for the case of the Blaschke product B_1 with the zeros $z_n = n + iy_n$, $n \in \mathbb{N}$.

Theorem 6.1. *Let $\{y_n\}_{n\in\mathbb{N}}$ be a positive nonincreasing sequence.*

1. If

$$\sum_{n\in\mathbb{N}} n^{-3/2}\log\frac{1}{y_n} < \infty, \tag{40}$$

then any even majorant w nonincreasing on $\mathbb{R}_+$ with convergent integral (5) is admissible for K_{B_1}.

2. Let $y : [0,\infty) \to (0,\infty)$ be a nonincreasing function such that $y(n) = y_n$, $n \in \mathbb{N}$, and let $Y = -\log y$. If the function $Y(e^x)$ is convex on $\mathbb{R}$ and the series (40) diverges, then any majorant w such that $w(t) = o(|t|^{-N})$, $t \to \infty$, for every $N > 0$, is not admissible for K_{B_1}.

Proof. The proof is analogous to the proof of Theorem 1.5. To prove Statement 1 we define the Blaschke product $B^\#$ as above, that is, $B^\#$ is the product with the

zeros $n + i(y_n + 1)$, $n \in \mathbb{N}$. By Theorem 1.3, each even majorant nonincreasing on $\mathbb{R}_+$ with convergent integral (5) is admissible for $K_{B^\#}$. Now we define w_1 by formula (37) and complete the proof as above.

To prove Statement 2 we use the same idea as in the proofs of Lemmas 3.5 and 3.6. First we note that if

$$f(t) = \sum_{n\in\mathbb{N}} \frac{\sqrt{y_n}c_n}{t - n + iy_n}$$

is a function from K_B such that for any N we have $|f(t)| = o(t^{-N})$, $t \to \infty$, then

$$\sum_{n\in\mathbb{N}} \sqrt{y_n}c_n(n - iy_n)^k = 0, \qquad k \in \mathbb{Z}_+.$$

Consider the function

$$g(t) = \sum_{n\in\mathbb{N}} \frac{\sqrt{y_n}c_n}{t - n + i(y_n + 1)}.$$

Clearly, $g \in K_{B^\#}$. On the other hand,

$$(t + i)^k g(t) - \sum_{n\in\mathbb{N}} \frac{(n - iy_n)^k\sqrt{y_n}c_n}{t - n + i(y_n + 1)} = 0.$$

Therefore,

$$|g(t)| \le C \inf_{k\in\mathbb{Z}_+} \frac{1}{|t + i|^k} \sup_{n\in\mathbb{N}}[\sqrt{y_n}|n - iy_n|^k].$$

Repeating the arguments from the proof of Lemma 3.6 we see that $|g(t)| \le t^A e^{-Y(t)}$, $t > 1$, where A is some positive constant. Since the series (40) diverges, we have $\int_1^\infty t^{-3/2}Y(t)dt = \infty$ and it follows from Theorem 1.3 that $g \equiv 0$. ◯

Remark. To obtain another proof of Theorem 6.1, one may consider the function $F(x) = \int_{\mathbb{R}} f(t)e^{-itx}dt$, $x > 0$, and show that F extends to $\mathbb{C}^+ \cup \mathbb{R}$ with estimates on the derivatives. To complete the proof, one should apply a slightly modified version of the quasianalyticity theorem due to B.I. Korenblum [17]. On the other hand, making use of the idea of the proof of Theorem 6.1 one can give another proof of Theorem 1.5, Statement 2.

One more proof of this result can be obtained by using a result of M.M. Dzhrbashyan [19, Theorem 24] on weighted polynomial approximation on nowhere dense sets dividing the complex plane.

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A.D. Baranov:
Saint Petersburg State University,
Department of Mathematics and Mechanics,
28, Universitetski pr., St. Petersburg,
198504, Russia

E-mail: antonbaranov@netscape.net

A.A. Borichev:
Laboratoire d'Analyse et Géométrie,
Université Bordeaux 1,
351, Cours de la Libération,
33405 Talence, France

E-mail: Alexander.Borichev@math.u-bordeaux1.fr

V.P. Havin:
Saint Petersburg State University,
Department of Mathematics and Mechanics,
28, Universitetski pr., St. Petersburg,
198504, Russia

E-mail: havin@VH1621.spb.edu